On the
Kumaraswamy–Dagum–Log–Logistic sigmoid functions with applications to population dynamics

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This article is dedicated to 75th anniversary of
Professor, Sc. D. Svetoslav Markov

Abstract.

The Kumaraswamy–Dagum distribution is a flexible and simple model with applications to income and lifetime data.

We prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function $\tilde{h}_{t_0}(t)$ by a class of Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF). Numerical examples, illustrating our results are given.

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1. Introduction. Dagum (1977) motivates his model from the empirical observation that the income elasticity \( \eta(F,t) \) of the cumulative distribution function (CDF) \( F \) of income is a decreasing and bounded function \( F \).

The cumulative distribution function (cdf) of Dagum distribution is given by
\[
G_D(t,\lambda,\beta,\delta) = (1 - \lambda t^{-\delta})^{-\beta},
\]
for \( t \geq 0 \), where \( \lambda \) is a scale parameter; \( \delta \) and \( \beta \) are shape parameters.

The cumulative distribution function (cdf) of Kumuraswamy distribution is given by
\[
G(t) = 1 - (1 - t^\psi)^\phi, \quad t \in (0,1)
\]
for \( \psi > 0 \) and \( \phi > 0 \).

This approach was further developed in a series of papers on generating systems for income distribution [4]–[7].

For other results, see [8], [9], [10].

For an arbitrary (cdf) \( F(t) \) with (PDF) \( f(t) = \frac{dF(t)}{dt} \), the family of Kumaraswamy–G distribution with (cdf) \( G_k(t) \) is given by
\[
G_k(t) = 1 - (1 - F^\psi(t))^\phi,
\]
for \( \psi > 0 \) and \( \phi > 0 \).

By letting \( F(t) = G_D(t) \), we obtain the Kumaraswamy–Dagum (KD) distribution, with (cdf)
\[
G_{KD}(t) = 1 - \left(1 - G_D^\psi(t)\right)^\phi,
\]
i.e.
\[ G_{KD}(t) = 1 - \left( 1 - \left( 1 + \lambda t^{-\delta} \right)^{-\beta} \right)^{\phi}. \] (5)

See [10] for further details.

When \( \beta = 1 \), we obtain Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF):

\[ G_{KD}(t) = 1 - \left( 1 - \left( 1 + \lambda t^{-\delta} \right)^{-1} \right)^{\phi}. \] (6)

In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function \( \tilde{h}_{t_0}(t) \) by a class of Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF).

2. Preliminaries.

**Definition 1.** The (basic) step function is:

\[ \tilde{h}_{t_0}(t) = \begin{cases} 
0, & \text{if } t < t_0, \\
[0, 1], & \text{if } t = t_0, \\
1, & \text{if } t > t_0,
\end{cases} \] (7)

usually known as shifted Heaviside function.

**Definition 2.** [12], [13] The Hausdorff distance (the H–distance) \( \rho(f, g) \) between two interval functions \( f, g \) on \( \Omega \subseteq \mathbb{R} \), is the distance between their completed graphs \( F(f) \) and \( F(g) \) considered as closed subsets of \( \Omega \times \mathbb{R} \).

More precisely,

\[ \rho(f, g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B|| \}, \] (8)

wherein \(||.||\) is any norm in \( \mathbb{R}^2 \), e. g. the maximum norm \(||(t, x)|| = \max\{|t|, |x|\}; hence the distance between the points \( A = (t_A, x_A), B = (t_B, x_B) \) in \( \mathbb{R}^2 \) is \(||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)\).
Let us point out that the Hausdorff distance is a natural measuring criteria for the approximation of bounded discontinuous functions \[1\].

3. Main Results.

Let us consider the following five parametric sigmoid function

\[
F^*(t) = 1 - \left( 1 - \left( (1 + \lambda t^{-\delta})^{-\beta} \right)^\phi \right)
\]

with

\[
F^*(t_0) = \frac{1}{2}, \quad t_0 = \left( \frac{1}{\lambda} \left( \left( 1 - 0.5^{\frac{1}{\psi}} \right)^{-\frac{1}{\lambda \psi}} - 1 \right) \right)^{-\frac{1}{\delta}}.
\]

The H-distance \( d = \rho(\tilde{h}_{t_0}, F^*) \) between the shifted Heaviside step function \( \tilde{h}_{t_0} \) and the sigmoidal function \( F^* \) satisfies the relation:

\[
F^*(t_0 + d) = 1 - \left( 1 - \left( (1 + \lambda(t_0 + d)^{-\delta})^{-\beta} \right)^\phi \right) = 1 - d.
\]

The following theorem gives upper and lower bounds for \( d \) in the case \( \beta = 1 \)

**Theorem 1.** Let

\[
a = - \left( 1 - \left( \frac{1}{1 + \left( \left(\frac{-1+(1-0.5^{\frac{1}{\psi}})}{\lambda}\right)^{-\frac{1}{\lambda \psi}} \right)^{-\delta} \right) \right)^\phi
\]

(12)
\[
b = 1 + \delta \left( \left( \frac{-1+(1-0.5 \phi) \frac{1}{\lambda} - \frac{1}{\lambda \psi}}{-1+(1-0.5 \phi) \frac{1}{\lambda} - \frac{1}{\lambda \psi}} \right)^{-\frac{1}{\delta}} \right)^{-1-\delta} \lambda \left( \frac{1}{1+\left( \left( \frac{-1+(1-0.5 \phi) \frac{1}{\lambda} - \frac{1}{\lambda \psi}}{-1+(1-0.5 \phi) \frac{1}{\lambda} - \frac{1}{\lambda \psi}} \right)^{-\frac{1}{\delta}}} \right)^{-1-\delta} \right) ^{1+\psi} \times \]

\[
\times \left( 1 - \left( \frac{1}{1+\left( \left( \frac{-1+(1-0.5 \phi) \frac{1}{\lambda} - \frac{1}{\lambda \psi}}{-1+(1-0.5 \phi) \frac{1}{\lambda} - \frac{1}{\lambda \psi}} \right)^{-\frac{1}{\delta}}} \right)^{-1+\phi} \right) ^{1+\psi} \right) ^{-1+\phi}. \tag{13}
\]

The H-distance \(d\) between the function \(\tilde{h}_{t_0}\) and the function \(F^*\) can be expressed in terms of the parameters for \(\frac{2b}{-a} > e^2\) as follows:

\[
d_l = \frac{1}{\frac{2b}{-a}} < d < \frac{\ln \left( \frac{2b}{-a} \right)}{\frac{2b}{-a}} = d_r. \tag{14}\]

**Proof.** We define the functions

\[
H(d) = F^*(t_0 + d) - 1 + d \tag{15}
\]

\[
G(d) = a + bd. \tag{16}
\]

From Taylor expansion

\[
H(d) - G(d) = O(d^2)
\]

we see that the function \(G(d)\) approximates \(H(d)\) with \(d \to 0\) as \(O(d^2)\) (cf. Fig. 1).

In addition \(G''(d) > 0\) and for \(\frac{2b}{-a} > e^2\)

\[
G(d_l) < 0; \quad G(d_r) > 0.
\]
This completes the proof of the inequalities (14).

The generated sigmoidal functions $F^*(t)$ for $\lambda = 0.1; \delta = 2.5; \beta = 1; \psi = 0.7; \phi = 1.8$ and $\lambda = 0.001; \delta = 3.5; \beta = 1; \psi = 0.8; \phi = 1.9$ are visualized on Fig. 2–Fig. 3.

From the Fig. 2–Fig.3 it can be seen that the "supersaturation" is fast.

Following Dagum (1977), in a period when individual data were rarely available, minimized

$$
\sum_{i=1}^{n} \left( F_n(t_i) - \left( 1 - \left( 1 - \left( 1 + \lambda t_i^{-\delta} \right)^{-\beta} \right)^\psi \right)^\phi \right)^2.
$$

a non–linear least–squares criterion based on the distance between the empirical $F_n$ and the CDF of a Kumaraswamy–Dagum approximation.

The appropriate least–square fitting of the real data (the experimental data - biomass for Xantobacter autotrophycum with electric field, see [26]) by the Dagum model yields for $\beta = 1, \lambda = 110, \delta = 1.45, \psi = 1.35$ and $\phi = 1.1$ and is visualized on Fig. 4.
Figure 2: The function $F^*(t)$ for $\lambda = 0.1; \delta = 2.5; \beta = 1; \psi = 0.7; \phi = 1.8; t_0 = 0.226373; H$-distance $d = 0.175123; d_l = 0.0689168; d_r = 0.184343$.

Figure 3: The function $F^*(t)$ for $\lambda = 0.001; \delta = 3.5; \beta = 1; \psi = 0.8; \phi = 1.9; t_0 = 0.0979526; H$-distance $d = 0.0763243; d_l = 0.0244187; d_r = 0.0906522$. 
The appropriate least–square fitting of the real data by the Dagum model yields for $\beta = 1$, $\lambda = 110$, $\delta = 1.45$, $\psi = 1.35$ and $\phi = 1.1$.

4. Conclusion

In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function $\tilde{h}_t(t)$ by a class of Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF).

A family of five parametric sigmoidal functions based on Kumurawamy–Dagum cumulative distribution function is introduced finding application in population dynamics.

Numerical examples, illustrating our results are given.

We propose a software module (intellectual property) within the programming environment CAS Mathematica for the analysis of the considered family of (KD–CDF) functions.

For other results, see [14]–[26].

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